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One-dimensional models with $1/r^2$ interactions

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Abstract. The critical behaviour of a general discrete one-dimensional model with inverse square interactions is discussed, using renormalisation group methods.

1. Introduction

The one-dimensional Ising model with long-range ferromagnetic interactions which decay asymptotically as $1/r^2$ is of considerable theoretical interest. Ising models with $1/r^{1+\sigma}$ interactions are known rigorously to exhibit long-range order if $\sigma < 1$, and have no transition if $\sigma > 1$ (Ruelle 1969, Dyson 1969). Thus the $1/r^2$ model stands on a dividing line. No rigorous information is known about the existence of long-range order in this case, but Thouless (1969) has argued that it has a finite T_c, for much the same reasons as has the two-dimensional XY model (Kosterlitz and Thouless 1973). Both models exhibit non-trivial topological defects which interact logarithmically. The energy of a single topological defect also diverges logarithmically with the size of the system, and at low temperatures the appearance of such defects is disfavoured. In the one-dimensional Ising model, these defects are just the domain walls. At higher temperatures the entropy term in the free energy dominates, and the domain walls appear, thus destroying the long-range order.

The $1/r^2$ Ising model is also of interest because it can be mapped into a path integral formulation of the spin-$\frac{1}{2}$ Kondo problem (Anderson and Yuval 1970) in which the one dimension row represents imaginary time, and the Ising spins represent the time history of the single impurity spin.

Anderson *et al* (1970), and Anderson and Yuval (1971), applied a primitive version of the renormalisation group to the $1/r^2$ Ising model, and obtained the expected phase structure, with its implications for the Kondo problem.

In this paper we consider a general discrete one-dimensional model with $1/r^2$ interactions. By discrete, we mean that there are a finite number of states available at each site. It is clear that the general considerations of Thouless (1969) hold also for these models. On the other hand, for models with a continuum of states we expect that $T_c \to 0$ as $\sigma \to 1$ from below (Kosterlitz 1976).

These general discrete models are of interest in view of their relationship to higher spin generalisations of the Kondo problem. It turns out that they also have features which bear a striking resemblance to higher-dimensional problems, such as two-dimensional melting, and quark confinement.

The layout of this paper is as follows. In § 2 we define the set of models under consideration, and show that their Hamiltonians may be exactly rewritten in terms of

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degrees of freedom describing the kinks and their logarithmic interactions. In § 3 it is shown that a suitable generalisation of the Anderson *et al* renormalisation scheme can be carried out, and that, remarkably, no new couplings are generated. The consequences for the critical behaviour are discussed in § 4, and a set of graphical rules is given by which one can determine the phase diagram of a given model. Two examples are discussed in detail: the Q-state Potts model and the Ashkin–Teller model. The Appendix is devoted to generalising the renormalisation group equations to the case of a general force law between the kinks, thus justifying some simplifications made in the text.

2. Formulation

We consider a one-dimensional chain, whose sites are labelled by integers n. At each site n is a variable σ_n which can be in one of S states $\alpha, \beta, \ldots$. The Hamiltonian is defined by

$$-\beta H = \sum_{m<n} V(n-m) K(\sigma_m, \sigma_n) \tag{2.1}$$

where $V(n) \sim 1/n^2$ as $n \to \infty$, and the interaction is ferromagnetic, i.e.

$$K(\alpha, \beta) = K(\beta, \alpha) < K(\alpha, \alpha). \tag{2.2}$$

Further restrictions on $V(n)$ will emerge in the course of our analysis. In principle, it is necessary that the interaction between sites m and n have the factorising form (2.1) only asymptotically as $(n-m) \to \infty$.

We now wish to rewrite (2.1) in terms of degrees of freedom describing the domain walls, or kinks. A kink of type $(\alpha\beta)$, with $\alpha \neq \beta$, is said to occupy the site n if $\sigma_n = \alpha$ and $\sigma_{n+1} = \beta$. Defining a new function $U(n)$ such that

$$V(n) = U(n+1) - 2U(n) + U(n-1) \tag{2.3}$$

the Hamiltonian, after a rearrangement of terms, can be written

$$-\beta H = \sum_{m<n} U(n-m)\{K(\sigma_{m+1}, \sigma_n) + K(\sigma_m, \sigma_{n+1}) - K(\sigma_m, \sigma_n) - K(\sigma_{m+1}, \sigma_{n+1})\}$$
$$+ \sum_n U(0)[K(\sigma_n, \sigma_{n+1}) - K(\sigma_n, \sigma_n)] + \sum_n (U(0) - U(1)) K(\sigma_n, \sigma_n). \tag{2.4}$$

Now observe that the expression in the curly brackets vanishes whenever $\sigma_n = \sigma_{n+1}$ or $\sigma_m = \sigma_{m+1}$. Thus this term is non-zero only when there are kinks at sites m and n, and therefore it describes the interaction between the two kinks. The second term in (2.4) gives a chemical potential for the kinks. From (2.3) one can show that

$$U(0) - U(1) = \sum_1^\infty V(r) \tag{2.5}$$

a quantity which we assume to be positive. The third term then shows that neighbouring kinks of types $(\alpha\beta)$ and $(\beta\gamma)$ attract each other with a linear potential proportional to $-K(\beta, \beta)$. We can use the arbitrariness in the choice of energy scale to impose the condition

$$\max_\alpha K(\alpha, \alpha) = 0. \tag{2.6}$$

States with a negative value of $K(\alpha, \alpha)$ will then be energetically disfavoured. In fact, since the important interactions which determine T_c are logarithmic rather than linear, we expect such states to be irrelevant even at finite temperatures. This can be made quantitative using the renormalisation group ideas developed in the Appendix, where we consider a general force law between kinks. The effect is that, in considering the critical behaviour, we can ignore all states α with $K(\alpha, \alpha) < 0$. For example, in the spin-1 Ising model where $\sigma_n = 0, \pm 1$ and a simple interaction $K(\sigma, \sigma') \propto \sigma\sigma' - 1$, the state $\sigma = 0$ will be irrelevant, and the critical behaviour will be that of the spin-$\frac{1}{2}$ Ising model.

The explicit solution of (2.3) is

$$U(n) = \sum_{r=1}^{n-1} (n-r)V(r) - n \sum_{r=1}^{\infty} V(r) + C \tag{2.7}$$

where we have imposed the boundary condition that $U(n)$ should contain no linear term as $n \to \infty$. C is chosen so that U contains no constant term either, that is

$$U(n) = -\ln n + O(1/n) \qquad \text{as } n \to \infty. \tag{2.8}$$

If $V(r) = 1/r^2$ for all $r \geq 2$, $U(0) = C = 1 + \gamma + V(1)$, where $\gamma \approx 0.577$ is Euler's constant.

3. Renormalisation

We now replace the original system, of kinks which occupy sites of a discrete lattice, by a continuous one where kinks are free to move on the line, but have a hard repulsive core of size a. Such a modification of the ultraviolet cut-off should not change the critical behaviour. We shall also approximate $U(r/a)$ by its asymptotic form $-\ln(r/a)$ for all $r > a$. As shown in the Appendix, this approximation is justified, since the fixed point form of the interaction must be scale invariant. Instead of using the chemical potential we introduce the fugacity $y_{\alpha\beta} = y_{\beta\alpha}$ for the kink $(\alpha\beta)$, which initially has the value

$$y_{\alpha\beta} = \exp[U(0)K(\alpha, \beta)]. \tag{3.1}$$

The partition function now takes the form

$$Z = \sum_{n=0}^{\infty} \sum_{\alpha,\beta} y_{\alpha_1\beta_1} y_{\alpha_2\beta_2} \ldots y_{\alpha_n\beta_n} \int (\mathrm{d}r_1\, \mathrm{d}r_2 \ldots \mathrm{d}r_n / a^n) \prod_j \theta(r_{j+1} - r_j - a)$$

$$\times \prod_{i<j} [(r_j - r_i)/a]^{K(\alpha_i,\alpha_j)+K(\beta_i,\beta_j)-K(\alpha_i,\beta_j)-K(\beta_i,\alpha_j)} \tag{3.2}$$

where the kink at position r_i is of type $(\alpha_i\beta_i)$ and we impose the constraint that the kinks are correctly ordered, i.e. $\beta_i = \alpha_{i+1}$. Periodic boundary conditions also imply that $\beta_n = \alpha_1$.

We now replace a by $a\mathrm{e}^l$ (with $l \ll 1$) and see how the various parameters $y_{\alpha\beta}$ and $K(\alpha, \beta)$ must transform in order to maintain the form of Z. The fundamental length a appears raised to a power in (3.2), and also in the theta functions controlling the short distance cut-off. If we neglect terms $O(l^2)$ the effects of these two dependences are

additive. In each term of (3.2), a appears raised to the power

$$-n-\sum_{i<j}[K(\alpha_i,\alpha_j)+K(\beta_i,\beta_j)-K(\alpha_i,\beta_j)-K(\beta_i,\alpha_j)] \tag{3.3}$$

$$=-n-\tfrac{1}{2}\sum_{i\neq j}[K(\alpha_i,\alpha_j)+K(\alpha_{i+1},\alpha_{j+1})-K(\alpha_i,\alpha_{j+1})-K(\alpha_{i+1},\alpha_j)] \tag{3.4}$$

$$=-n-\sum_i K(\alpha_i,\beta_i). \tag{3.5}$$

The change $a\to ae^l$ can therefore be compensated by making $y_{\alpha\beta}\to y_{\alpha\beta}\exp(1+K(\alpha,\beta))l$. The fact that such a simple renormalisation is possible, while trivial for the Ising model, is quite remarkable in this general case.

Turning to the effect of changing a in the cut-off, once again the neglect of $O(l^2)$ terms enables us to consider each θ-function separately. Writing

$$\theta(r_{j+1}-r_j-ae^l)=\theta(r_{j+1}-r_j-a)-al\delta(r_{j+1}-r_j-a)+O(l^2) \tag{3.6}$$

we see that the additional effect is to juxtapose each neighbouring kink pair in turn. The factors involving two such kinks i, $i+1$ and a third kink j are

$$ly_{\alpha_i\beta_i}y_{\alpha_{i+1}\beta_{i+1}}[(r_j-r_i)/a]^{K(\alpha_i,\alpha_j)+K(\beta_i,\beta_j)-K(\alpha_i,\beta_j)-K(\beta_i,\alpha_j)}$$
$$\times[(r_j-r_i-a)/a]^{K(\alpha_{i+1},\alpha_j)+K(\beta_{i+1},\beta_j)-K(\alpha_{i+1},\beta_j)-K(\beta_{i-1},\alpha_j)}. \tag{3.7}$$

For $r_j-r_i\gg a$ the second bracket may be approximated by (r_j-r_i/a). This neglects terms which correspond to $1/r$ interactions between the kinks, which are irrelevant in the renormalisation group sense. Recalling that $\beta_i=\alpha_{i+1}$, (3.7) reduces to

$$ly_{\alpha_i\beta_i}y_{\beta_i\beta_{i+1}}[(r_j-r_i)/a]^{K(\alpha_i,\alpha_j)+K(\beta_{i+1},\beta_j)-K(\alpha_i,\beta_j)-K(\beta_{i+1},\alpha_j)}. \tag{3.8}$$

As long as $\alpha_i\neq\beta_{i+1}$, this can be incorporated into a renormalisation of $y_{\alpha_i\beta_{i+1}}$. In general, this will take the form

$$y_{\alpha\beta}\to y_{\alpha\beta}+l\sum_\gamma y_{\alpha\gamma}y_{\gamma\beta}. \tag{3.9}$$

However, if $\alpha_i=\beta_{i-1}$, the pair is neutral, and (3.8) is independent of (r_j-r_i). This leading term then contributes only a constant to the free energy, and one must proceed to the next term in the expansion of (3.7), which is

$$ly^2_{\alpha_i\beta_i}[K(\alpha_i,\alpha_j)+K(\beta_i,\beta_j)-K(\alpha_i,\beta_j)-K(\beta_i,\alpha_j)]a/(r_j-r_i). \tag{3.10}$$

Integrating over the allowed range $r_{i-1}\leqslant r_i\leqslant r_{i+2}$, we obtain

$$ly^2_{\alpha_i\beta_i}[K(\alpha_i,\alpha_j)+K(\beta_i,\beta_j)-K(\alpha_i,\beta_j)-K(\beta_i,\alpha_j)]\{\ln[(r_j-r_{i-1})/a]-\ln[(r_j-r_{i+2})/a]\}. \tag{3.11}$$

These two terms may be interpreted as a renormalisation of the interaction of a kink at r_j with those at r_{i-1}, r_{i+2} respectively, if we recall that $\beta_{i-1}=\alpha_i=\alpha_{i+2}$. After correctly symmetrising by including the effects of neutral pairs to the left of r_{i-1} and the right of r_{i+2}, terms like (3.11) can be expressed as a renormalisation of the couplings $K(\alpha,\beta)$:

$$K(\alpha,\beta)\to K(\alpha,\beta)-l\sum_\gamma y^2_{\alpha\gamma}(K(\alpha,\beta)+K(\alpha,\gamma)-K(\beta,\gamma))$$
$$-l\sum_\gamma y^2_{\beta\gamma}(K(\alpha,\beta)+K(\beta,\gamma)-K(\alpha,\gamma)). \tag{3.12}$$

Note that this preserves the condition $K(\alpha,\alpha)=0$.

Combining all three contributions we obtain the renormalisation group equations ($K_{\alpha\beta} = -K(\alpha, \beta) > 0$):

$$\mathrm{d}K_{\alpha\beta}/\mathrm{d}l = -\sum_{\gamma} y_{\alpha\gamma}^2(K_{\alpha\beta} + K_{\alpha\gamma} - K_{\beta\gamma}) - \sum_{\gamma} y_{\beta\gamma}^2(K_{\alpha\beta} + K_{\beta\gamma} - K_{\alpha\gamma}) \tag{3.13}$$

$$\mathrm{d}y_{\alpha\beta}/\mathrm{d}l = (1 - K_{\alpha\beta})y_{\alpha\beta} + \sum_{\gamma} y_{\alpha\gamma}y_{\beta\gamma}. \tag{3.14}$$

These equations are the central result of this paper. The only approximations that have been made are the neglect of $1/r$ interactions between the kinks, which are irrelevant by dimensional counting, and the neglect of excluded volume effects. These non-universal terms also appear in the XY model (Cardy and Parga 1980). They are difficult to evaluate, but fortunately appear only in $\mathrm{O}(y^4)$. If $U(0)$ is large enough (which can be guaranteed by, for example, making $V(1)$ large) the initial value of $y_{\alpha\beta}$ is small even at T_c, and the equations should be valid.

4. Solution

Equations (3.13, 3.14) exhibit an $S(S-1)/2$-dimensional space of fixed points $y_{\alpha\beta} = 0$. At low temperatures all the $K_{\alpha\beta}$ will be large and this fixed subspace will be attractive. The first phase transition will be characterised by some of the $y_{\alpha\beta}$ becoming marginal, when one or more of the $K_{\alpha\beta}$ becomes equal to unity. If there is no underlying symmetry in the model, in general there will be a unique largest $K_{\alpha\beta}$, with $(\alpha\beta) = (\bar{\alpha}\bar{\beta})$, say. At the critical point, all the $y_{\alpha\beta} \neq y_{\bar{\alpha}\bar{\beta}}$ will remain irrelevant. In this case the equations reduce to those describing the Ising model, and one obtains the usual results, with for example the correlation length ξ diverging as $T \to T_c +$ according to

$$\xi \sim \xi_0 \exp[b(T - T_c)^{-p}] \tag{4.1}$$

with $p = \frac{1}{2}$, and ξ_0, b non-universal numbers.

When several of the $K_{\alpha\beta}$ become critical at the same time, the transition can be characterised graphically as follows. Draw a graph whose vertices are labelled by the states $\alpha, \beta, \ldots$. A particular transition corresponds to drawing a set of edges $(\alpha\beta)$ corresponding to those $K_{\alpha\beta}$ which become critical. The universality class of the transition then depends only on the topology of the graph. Note that if $K_{\alpha\beta}$ and $K_{\beta\gamma}$ become critical, then the second term in (3.14) will cause $y_{\alpha\gamma}$ to become relevant above the transition temperature, even though $K_{\alpha\gamma}$ is not critical. Physically, this is because, above T_c, the $(\alpha\beta)$ and $(\beta\gamma)$ kinks form a plasma in which screening occurs. Not only do these kinks screen the logarithmic interaction between kinks of their own kind, but also between the $(\alpha\gamma)$ kinks, which may be viewed as bound pairs of the $(\alpha\beta)$ and $(\beta\gamma)$ kinks. Thus, above T_c, all the kinks associated with states at which edges of a connected graph terminate form a plasma, and no longer have logarithmic interactions amongst themselves. They still, however, interact logarithmically with kinks in a disconnected piece of the graph. To investigate the possibility of further transitions, one should then shrink the connected pieces of the graph down to single vertices, and then repeat the procedure. If all states are connected no further transitions are possible, and the high-temperature phase has been reached.

In this way it is possible to classify all possible transitions which can occur in a given model. We now illustrate with two simple examples.

4.1. Q-state Potts model

This model has the maximum amount of symmetry, and undergoes only one phase transition. The couplings are given by

$$K_{\alpha\beta} = K(1-\delta_{\alpha\beta}). \tag{4.2}$$

In this case $y_{\alpha\beta} = y(\alpha \neq \beta)$ and the renormalisation group equations become

$$\mathrm{d}K/\mathrm{d}l = -2QKy^2 \tag{4.3}$$

$$\mathrm{d}y/\mathrm{d}l = (1-K)y + (Q-2)y^2. \tag{4.4}$$

For $Q \neq 2$ we note the appearance of a y^2 term in the second equation, which is forbidden in the Ising case because of the $y \to -y$ symmetry of the partition function. Its appearance in general is a consequence of the fact that three kinks can bind to form a neutral object. A similar situation, and similar equations, arise in the problem of the melting of a triangular lattice (Nelson 1978). The reader is referred to this article for a diagram of the renormalisation flows. There are two separatrices $y = m^{(\pm)}x$, where $x = 1-K$ and

$$m^{(\pm)} = \{(Q-2) \pm [(Q-2)^2 + 8Q]^{1/2}\}/4Q. \tag{4.5}$$

Putting $m^{(-)} = -m_1$ and $m^{(+)} = m_2$, the renormalisation group trajectories are

$$(y + m_1 x)^{m_1}(y - m_2 x)^{m_2} = t^{m_1} \tag{4.6}$$

where we have imposed the initial condition that $y = -m_1 x + \mathrm{O}(t)$ for $x = \mathrm{O}(-1)$. The parameter t is thus proportional to $T - T_c$. To evaluate the correlation length ξ, we integrate the equations out to $x = \mathrm{O}(+1)$, where the correlation length is finite, to obtain

$$\ln \xi \sim \int_{-1}^{1} \frac{\mathrm{d}x}{\mathrm{d}x/\mathrm{d}l} = \int_{-1}^{1} \frac{\mathrm{d}x}{2Qy^2}. \tag{4.7}$$

The asymptotic behaviour as $t \to 0$ may be evaluated by a simple rescaling $y \to yt^{m_1/(m_1+m_2)}$, $x \to xt^{m_1/(m_1+m_2)}$ which makes the constraint (4.6) independent of t. In this way

$$\ln \xi \sim t^{-m_1/(m_1+m_2)}[1 + \mathrm{O}(t^{m_1/(m_1+m_2)})]. \tag{4.8}$$

We conclude that the correlation length diverges much as in the Ising model, but with an index

$$p = \tfrac{1}{2}\{1 - (Q-2)/[(Q-2)^2 + 8Q]^{1/2}\}. \tag{4.9}$$

4.2. Ashkin–Teller model

This model has four states related by a Z_4 symmetry. If we label them 1, 2, 3, 4, the couplings are $K_1 = K_{12} = K_{23} = K_{34} = K_{41}$ and $K_2 = K_{13} = K_{24}$. The RG equations reduce to

$$\mathrm{d}K_1/\mathrm{d}l = -2(4K_1 - K_2)y_1^2 - 2K_2 y_2^2 \tag{4.10}$$

$$\mathrm{d}K_2/\mathrm{d}l = -4K_2 y_1^2 - 4K_2 y_2^2 \tag{4.11}$$

$$\mathrm{d}y_1/\mathrm{d}l = (1-K_1)y_1 + 2y_1 y_2 \tag{4.12}$$

$$\mathrm{d}y_2/\mathrm{d}l = (1-K_2)y_2 + 2y_1^2. \tag{4.13}$$

When $K_1 = K_2$ these reduce to the equations for the four-state Potts model. When $K_2 = 2K_1$ and $y_2 = y_1^2$ the model decouples into two Ising models, and the equations behave appropriately. We must distinguish two cases $K_1 \lessgtr K_2$. For $K_1 > K_2$ the first transition occurs when $K_2 = 1$. At this point y_2 becomes relevant, but not y_1. That is, kinks of the type (13) and (24) unbind, but they cannot screen the interaction between the other kinks, in the same way that integer electric charges cannot screen half-integer charges. The half-integer charges are then able to undergo a second transition when $K_1 = 1$. Graphically this sequence is represented in figure 1. When $K_1 < K_2$ the first transition occurs at $K_1 = 1$, where both y_1 and y_2 become relevant. The half-integer charges are capable of screening the integer charges, so no further phase transition occurs. This is represented in figure 2. The full phase diagram is shown schematically in figure 3. Our results are consistent with the limits $K_1 = \infty$ and $K_2 = 0$, when the model reduces to a single Ising model. Note that all the transitions are Ising-like (except at the Potts point), unlike the case of two dimensions and short-ranged interactions.

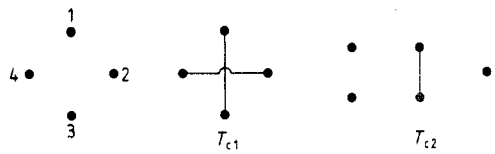

Figure 1. Graphs illustrating the sequence of transitions in the Ashkin–Teller model when $K_1 > K_2$.

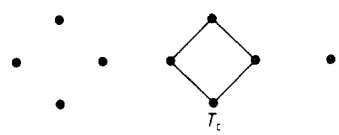

Figure 2. The same as figure 1, with $K_1 < K_2$.

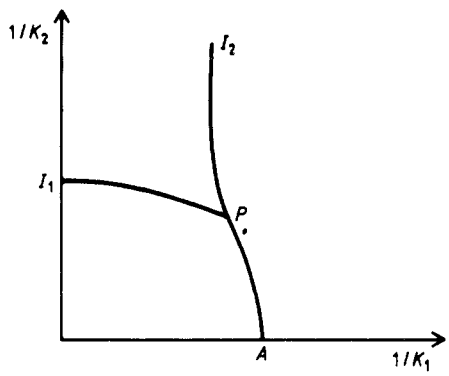

Figure 3. Schematic phase diagram of the Ashkin–Teller model. Across the boundary I_1P kinks (13) and (14) unbind. Across I_2PA kinks (12), (23), (34) and (41) unbind. P represents the four-state Potts model critical point.

5. Conclusions

We have shown that the general discrete one-dimensional model with $1/r^2$ interactions can be solved using a generalisation of the renormalisation scheme of Anderson *et al* (1970). Since in the S-state model there are $S(S-1)$ kinds of kink, and $O(S^4)$ interactions, it is remarkable that the renormalisation can be carried through without extending the parameter space beyond that of the original model, even though no symmetry is assumed.

In general these models show multiple phase transitions of the same general type as in the Ising model. That is, the correlation length ξ diverges like $\exp[b/(T_c-T)^p]$ with p however differing from its Ising value of $\frac{1}{2}$, the specific heat shows an essential singularity like ξ^{-1}, and the order parameter jumps discontinuously to zero. We have illustrated a general graphical scheme whereby the phase diagram of a particular model may be found. Apart from their relevance to generalisations of the Kondo problem, these models provide simple examples of the physics of topological phase transitions, charge confinement and screening.

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Appendix

Suppose that the interaction energy of the kinks is not simply proportional to $\ln(r/a)$, but is a general function $U(r/a)$. For simplicity we consider only the Ising case, so that the partition function is

$$Z=\sum_{n=0}^{\infty} y^{2n}\int (\mathrm{d}r_1\ldots\mathrm{d}r_{2n}/a^{2n})\prod_j(\theta_{j+1}-\theta_j-a)\prod_{i<j}\exp[(-1)^{j-i}U(r_j-r_i/a)]. \tag{A1}$$

In this expression there is an ambiguity in the definition of y, since we have not defined the zero of energy. So we impose the condition that $U(1)=0$. The renormalisation may be carried through as before. On replacing $a\to a\mathrm{e}^l$,

$$U(r/a\mathrm{e}^l)=U(r/a)-l(r/a)U'(r/a)+O(l^2). \tag{A2}$$

Hence $U(x)$ should be renormalised according to

$$U(x)\to U(x)+lxU'(x). \tag{A3}$$

However, this does not preserve the constraint $U(1)=0$. Thus we subtract a term $lU'(1)$ from the right-hand side of (A3), and compensate by renormalising y. The screening term coming from juxtaposing kinks is straightforward to evaluate. The resulting equations describe flows in the space of functions $U(x, l)$:

$$\partial U/\partial l=xU'(x)-U'(1)-4U(x)y^2 \tag{A4}$$

$$\mathrm{d}y/\mathrm{d}l=(1-U'(1))y \tag{A5}$$

where $U'(x) = \partial U/\partial x$. The fixed points occur at $y = 0$, $U(x) = K \ln x$. If we ignore the terms involving y, the solution is

$$U(x, l) = U(x e^l, 0) - U(e^l, 0) \tag{A6}$$

which shows that the fixed point can only be reached if the initial interaction $U(x, 0)$ is asymptotically logarithmic. Otherwise, the kinks will be either permanently bound or permanently free. Similar considerations may be applied to the two-dimensional XY model (Kosterlitz 1974). In this case the screening term is more complicated and involves a convolution of U with itself.

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